Massive Parallel Solutions of Variable Annuity PDEs

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April 2012

Outline

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- American Put Option
  - 5D, 6D
- Guaranteed Minimum Withdrawal Benefit (GMWB)

FITOB overview

- FITOB: developed research software at our Chair
- Modeling framework: by given a set of tradable and non-tradable assets, defined by stochastical differential equations (SDE):

$$\begin{aligned} \mathbf{S} &= \{S_1, S_2, \ldots, S_N\} \\ dS_i &= \mu_i(\mathbf{S}, t)\ dt + \sigma_i(\mathbf{S}, t)\ dW_i \\ dW_i dW_j &= \rho_{i,j}\ dt \end{aligned}$$

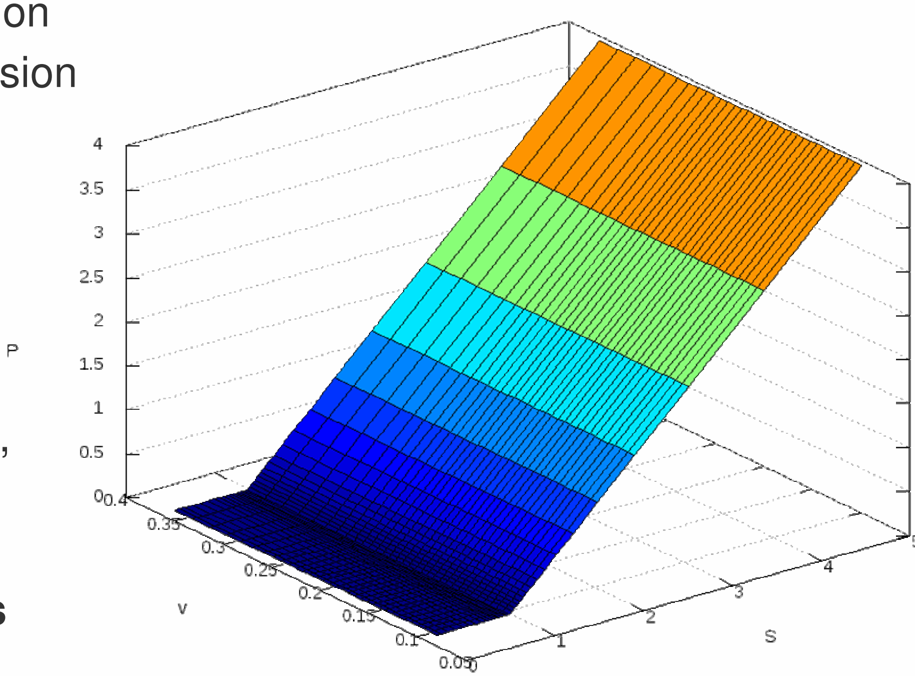
- The resulting Black-Sholes partial differential equation (BS-PDE)

$$\frac{\partial V}{\partial t} + \sum_{i=1}^{N} \hat{\mu}_i(\mathbf{S}, t) \frac{\partial V}{\partial S_i} + \frac{1}{2} \sum_{i,j=1}^{N} \sigma_i(\mathbf{S}, t) \sigma_j(\mathbf{S}, t) \rho_{i,j} \frac{\partial^2 V}{\partial S_i \partial S_j} - r(\mathbf{S}, t) V = 0$$

- Models defined by: $\mu_i(\mathbf{S}, t), \hat{\mu}_i(\mathbf{S}, t), \sigma_i(\mathbf{S}, t), r(\mathbf{S}, t)$ and $\rho_{i,j}$

FITOB overview

- Numerical aspects:
  - automatic computational domain estimation
  - Combination Technique for higher dimension
    - distributed memory parallelization
  - efficient Multigrid solvers
    - finite difference
    - adaptive time stepping
    - shared memory parallel
    - constraints enforcements (American, Barrier features)
  - hybrid parallelization
  - **no derivative or SDE specific methods** (e.g., log transformation)



FITOB overview

User interaction:

- Modeling Language ThetaML:
  - general and unique product description
  - Presented previously

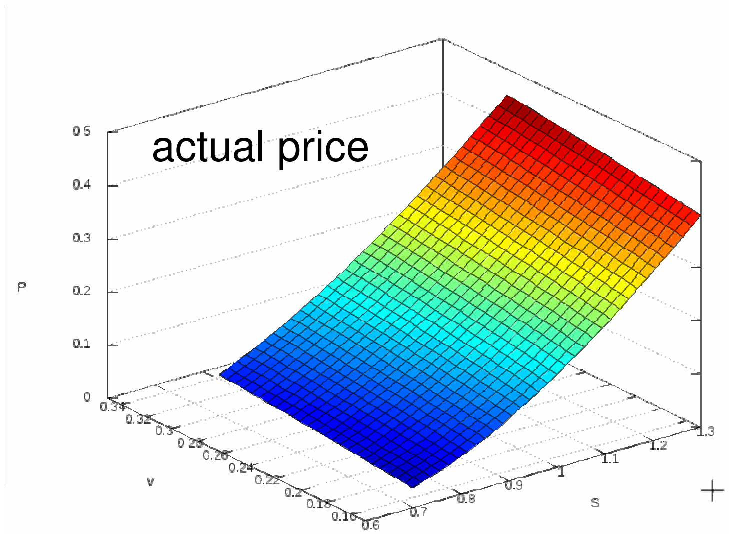
- XML configuration file:
  - SDE models
  - start values (and intervals)
  - solver parameters
  - could be coupled to GUI

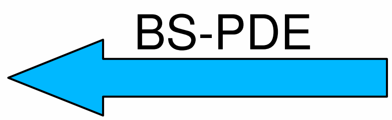
```
model ASIAN;
    import S;
    export P;

    A=0; N=12; K=0;
    loop (N)
       A=(K*A+S)/(K+1);
       P=MAX( P ,  A );
       Theta 1/N;
        K=K+1;
     end;
     P=MAX( A , 0.0 );
end;
```

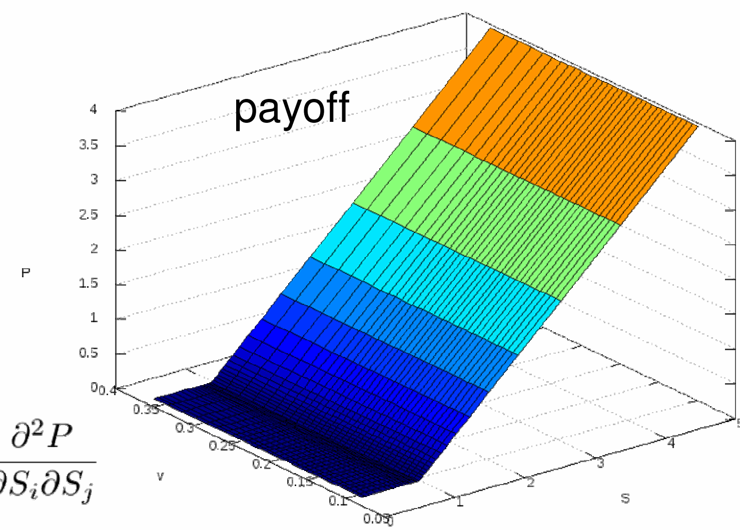
```
<variable name="S" evaluation="100" min="97" max="103" />
...
<variable name="S" type="GB" drift="0.05" convec="0.05" sigma="0.4"/>
...
<RISK_FREE_RATE value="0.05"/>
```

European Options

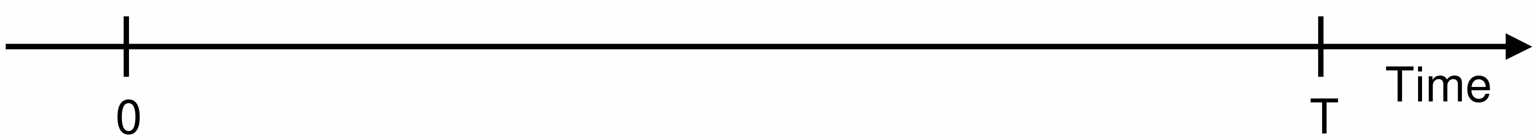
- BS-PDE based pricing (recapitulation)







$$\frac{\partial P}{\partial t} + \sum_{i=1}^{N} \hat{\mu}_i (\mathbf{S}, t) \frac{\partial P}{\partial S_i}$$
$$+ \quad \frac{1}{2} \sum_{i,j=1}^{N} \sigma_i (\mathbf{S}, t) \sigma_j (\mathbf{S}, t) \rho_{i,j} \frac{\partial^2 P}{\partial S_i \partial S_j}$$
$$- \quad r (\mathbf{S}, t) P = 0$$



European Options

- 5D example (N=5)

$$\begin{aligned} dS_i &= \mu_i S_i \, dt + \sigma_i S_i \, dW_i \\ \mu_i &= r = 0.05, \quad \sigma_i = 0.4, \quad S_i(0) = 1 \\ \rho_{i,j} &= 0.1, \; i \neq j \end{aligned}$$

```
model EUROPEAN5D;
    import S1;
    import S2;
    import S3;
    import S4;
    import S5;
    export P;

    Theta 1;
    P=MAX(S1+S2+S3+
        S4+S5-5.0 , 0.0 );
end;
```

- Results

level	price	rel. error	L_2	L_∞
3	0.530167	-3.801e-2	5.069e-2	2.906e-1
4	0.548944	-3.942e-3	4.119e-2	3.422e-1
5	0.550686	-7.829e-4	9.953e-4	3.203e-3
6	**0.551117**			

European Options

```
model EUROPEAN6D;
    import S1;
    import S2;
    import S3;
    import S4;
    import S5;
    import S5;
    export P;

    Theta 1;
    P=MAX(S1+S2+S3+
        S4+S5+S6-6.0 , 0.0 );
end;
```

- 6D example (N=6)

$$
\begin{aligned}
dS_i &= \mu_i S_i\, dt + \sigma_i S_i\, dW_i \\
\mu_i &= r = 0.05, \quad \sigma_i = 0.4, \quad S_i(0) = 1 \\
\rho_{i,j} &= 0.1, \; i \neq j
\end{aligned}
$$

- Results

level	price	rel. error	L_2	L_∞
3	0.614752	-3.095e-2	2.290e-2	8.979e-2
4	0.625327	-1.428e-2	5.244e-2	6.435e-1
5	0.634132	-4.047e-4	7.499e-4	1.979e-3
6	**0.634389**			

European Options

Parallel results for 5D and level = 6

- with regular grid number of points ~ 1.07e+9
- with combination technique number of points ~ 1.0e+7 (56 X 2.5e+5)

Runtimes

Number of processors	1	2	4	8	16	32	64	128
Total runtime (seconds)	8760	4080	3180	1638	966	465	259	240
Efficiency	1	1,07	0,69	0,67	0,57	0,59	0,53	0,29

2.5 hours

4 minutes

European Options

Parallel results for 6D and level = 6

- with regular grid number of points ~ 6.8e+10
- with combination technique number of points ~ 1.2e+8 (84 X 2e+6)

Runtimes

Number of processors	8	16	32	64	128
Total runtime (seconds)	24660	18060	7299	4881	3337
Efficiency	1	0,68	0,84	0,63	0,46

7 hours

55 minutes

American Options

- 5D example (N=5)

$$
\begin{aligned}
dS_i &= \mu_i S_i\, dt + \sigma_i S_i\, dW_i \\
\mu_i &= r = 0.05, \quad \sigma_i = 0.4, \quad S_i(0) = 1 \\
\rho_{i,j} &= 0.1, \; i \neq j
\end{aligned}
$$

- Results

level	price	rel. error	L_2	L_∞
3	0.303934	-8.597e-2	6.396e-2	2.842e-1
4	0.334562	6.137e-3	5.301e-2	4.113e-1
5	0.331521	-3.008e-3	1.660e-3	4.515e-2
6	**0.332522**			

```
model AMERICAN5D;
    import S1;
    import S2;
    import S3;
    import S4;
    import S5;
    export P;

    loop (inf)
      P = MAX(5.0-S1-S2-S3-
            S4-S5,P);
   end;


    Theta 1;
    P=MAX(5.0-S1-S2-S3-
        S4-S5, 0.0 );
end;
```

American Options

- 6D example (N=6)

$$
\begin{aligned}
dS_i &= \mu_i S_i \, dt + \sigma_i S_i \, dW_i \\
\mu_i &= r = 0.05, \quad \sigma_i = 0.4, \quad S_i(0) = 1 \\
\rho_{i,j} &= 0.1, \; i \neq j
\end{aligned}
$$

- Results

level	price	rel. error	L_2	L_∞
3	0.34839	-6.369e-2	3.064e-2	2.843e-1
4	0.35612	-4.292e-2	6.671e-2	6.843e-1
5	0.37118	-2.436e-2	1.854e-3	2.972e-2
6	**0.37209**			

```
model AMERICAN6D;
    import S1;
    import S2;
    import S3;
    import S4;
    import S5;
    import S6;
    export P;

    loop (inf)
      P = MAX(6.0-S1-S2-S3-
              S4-S5-S6,P);
    end;


    Theta 1;
     P=MAX(6.0-S1-S2-S3-
        S4-S5-S6, 0.0 );
end;
```

American Options

Parallel results for 5D and level = 6

- with regular grid number of points ~ 1.07e+9
- with combination technique number of points ~ 1.0e+7 (56 X 2.5e+5)

Runtimes

Number of processors	8	16	32	64	128
Total runtime (seconds)	4405	2282	1342	601	361
Efficiency	1	0,97	0,82	0,92	0,76

1.25 hours

6 minutes

American Options

Parallel results for 6D and level = 6

- with regular grid number of points ~ 6.8e+10
- with combination technique number of points ~ 1.2e+8 (84 X 2e+6)

Runtimes

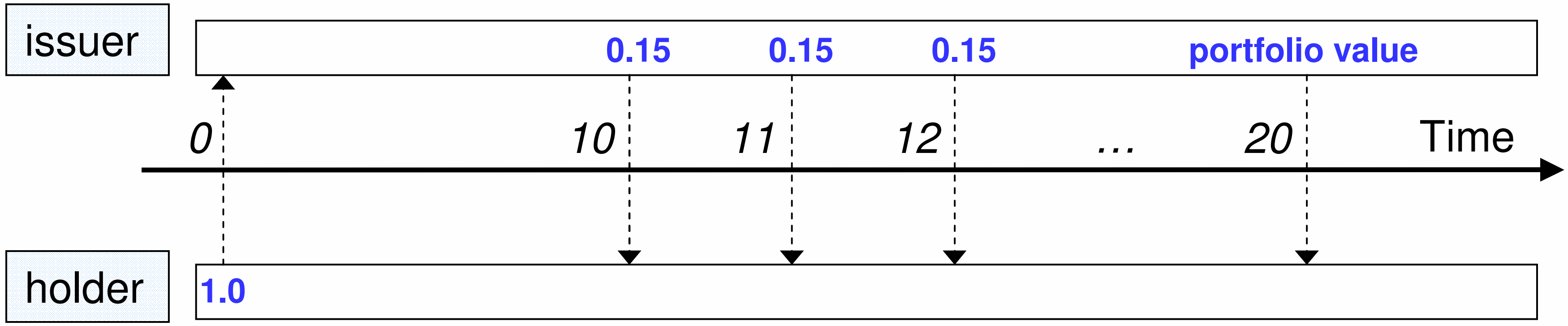
Number of processors	16	32	64	128	192
Total runtime (seconds)	37941	18144	10357	7594	6201
Efficiency	1	1,05	0,92	0,62	0,51

10.5 hours

1 hour 40minutes

Guaranteed Minimum Withdrawal Benefit (GMWB)

- Initial investment of 1.0 into an asset (forming a portfolio)
- We model the payment period as 10 years long
- In the payment period the portfolio structure stays the same (one asset)
- Withdrawal rights at each payment dates
- In case of no-withdrawal a fee of 2% is paid



Guaranteed Minimum Withdrawal Benefit (GMWB)

- ThetaML script
- todo

```
model GMWB5D;
    import S;
    export P;

    CI = 0.15;
    Theta 10;
    A=S;
    loop (10)
        ....
        Theta  1;
    end;
    P=MAX(A, 0.0 );
end;
```

Guaranteed Minimum Withdrawal Benefit (GMWB)

- todo

```
model GMWB5D;
    import S;
    export P;

    CI = 0.15;
    Theta 10;
    A=S;
    Sold = S;
    loop (10)
        P=MAX(P+CI,A);
        if (A<0)
            A = A - CI;
        else
            A = 0.98*(A-CI+
                  (A/SOLD)*(S-SOLD));
        end;
        Sold = S;
        Theta  1;
    end;
    P=MAX(A, 0.0 );
end;
```

Guaranteed Minimum Withdrawal Benefit (GMWB)

- Heston model of the asset

$$\begin{cases} dS(t) &= \mu\, S(t)\, dt + \sqrt{v(t)}dW_1 \\ dv(t) &= \kappa(\tilde{v} - v(t))dt + \sigma dW_2 \end{cases}$$

- CIR model of the interest rate

$$dr(t) = \lambda(\theta - r(t))dt + \sigma_r \sqrt{r(t)}dW_3.$$

- parameter and initial values

$$\mu = r(t)\ ,\ \kappa = 2.0\ ,\ \tilde{v} = 0.06\ ,$$
$$\sigma = 0.3\ ,\ \lambda = 3.0\ ,\ \theta = 0.05\ ,\ \sigma_r = 0.4$$
$$\rho_{1,2} = -0.5\ ,\ \rho_{1,3} = \rho_{2,3} = 0.0$$
$$V(0) = 0.05\ ,\ S(0) = 1.0\ ,\ r(0) = 0.01$$

```
model GMWB5D;
    import S;
    export P;

    CI = 0.15;
    Theta 10;
    A=S;
    Sold = S;
    loop (10)
        P=MAX(P+CI,A);
        if (A<0)
            A = A - CI;
        else
           A = 0.98*(A-CI+
                   (A/SOLD)*(S-SOLD));
        end;
        Sold = S;
        Theta  1;
    end;
    P=MAX(A, 0.0 );
end;
```

Guaranteed Minimum Withdrawal Benefit (GMWB)

- **Pricing results:**

level	price	rel. error	L_2	L_∞
3	2.0878	6.546e-1	9.006e-1	1.318e-0
4	1.8716	4.832e-1	7.404e-1	1.276e-0
5	1.3245	4.895e-2	7.534e-2	1.275e-1
6	1.2381	-1.880e-2	2.819e-2	4.727e-2
7	1.2514	-8.233e-3	1.086e-2	1.554e-2
8	1.2618			

- **Runtimes for level = 8**

Number of processors	4	16	32	64
Total runtime (seconds)	216000	85253	50871	40479
Efficiency	1	0,63	0,536	0,33

11 hours

Thank you!
Questions?

Literature:

- [1]
- [2]
- [3]
- [4]